

Why are quadratic normal volatility models analytically tractable?*

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Abstract

We discuss the class of “Quadratic Normal Volatility” (QNV) models, which have drawn much attention in the financial industry due to their analytic tractability and flexibility. We characterize these models as the ones that can be obtained from stopped Brownian motion by a simple transformation and a change of measure that only depends on the terminal value of the stopped Brownian motion. This explains the existence of explicit analytic formulas for option prices within QNV models in the academic literature. Furthermore, via a different transformation, we connect a certain class of QNV models to the dynamics of geometric Brownian motion and discuss changes of numéraires if the numéraire is modelled as a QNV process.

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1 Introduction and model

Quadratic Normal Volatility (QNV) models recently have drawn much attention in both industry and academia since they are not only easily tractable as generalizations of the standard Black-Scholes framework but also can be, due to their flexibility, well calibrated to various market scenarios. In this note, we focus on associating the dynamics of QNV processes to the dynamics of Brownian motion and geometric Brownian motion. These relationships reveal why analytic formulas can be (and indeed have been) found for option prices. However, we shall abstain here from computing explicit option prices implied by a QNV model. Formulas for these can be found for example in Andersen (2011).

More precisely, after introducing QNV models in this section and providing an overview of the relevant literature, we show in Section 2 how QNV models can be obtained from transforming a stopped Brownian motion. In Section 3 we work out a connection between a certain class of

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QNV processes and geometric Brownian motion and in Section 4 we formalize the observation that QNV models are stable under changes of numéraires. Section 5 contains some preliminary results on semi-static hedging within QNV models and Section 6 provides an interpretation of the strict local martingale dynamics of certain QNV processes as a possibility of a hyperinflation under a dominating measure, in the spirit of Carr et al. (2012). The appendix states an application of Föllmer's measure, which we shall use in some of the sections.

Model

If not specified otherwise, we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, equipped with a Brownian motion B . We introduce a process Y with deterministic initial value $Y_0 = y_0 > 0$, whose dynamics solve

$$dY_t = (e_1 Y_t^2 + e_2 Y_t + e_3) dB_t, \quad (1)$$

where $e_1, e_2, e_3 \in \mathbb{R}$. Problem 3.3.2 in McKean (1969) yields the existence of a unique, strong solution to this stochastic differential equation. We define S as the first hitting time of zero by Y and shall also study a stopped version X of Y , defined by $X := Y^S$.

Clearly, the dynamics of Y and X strongly depend on the parameters e_1, e_2, e_3 in the quadratic polynomial $P(z) := e_1 z^2 + e_2 z + e_3$ appearing in (1). We shall say that Y (X) is a (stopped) QNV process with polynomial P . The most important special cases are

- $e_1 = e_2 = 0$ corresponding to standard Brownian motion,
- $e_1 = e_3 = 0$ corresponding to geometric Brownian motion,
- $e_2 = e_3 = 0$ corresponding to the reciprocal of a three-dimensional Bessel process.

Feller's test for explosions directly yields that Y does not hit any real roots of P (except if $P(y_0) = 0$, in which case $Y \equiv y_0$ would just be constant); see Theorem 5.5.29 of Karatzas and Shreve (1991). The roots of P in (1) determine whether Y is a true \mathbb{Q} -martingale or a strict \mathbb{Q} -local martingale:

Proposition 1 (Martingality of QNV processes). *The QNV process Y is a true martingale if and only if $e_1 = 0$ or if P has two real roots $r_1 \leq r_2$ with $y_0 \in [r_1, r_2]$. The stopped QNV process X is a true martingale if and only if $e_1 = 0$ or P has a root r with $x_0 = y_0 \leq r$.*

The proposition is proved in Section 4.

Relevant literature

An incomplete list of authors who study QNV models in various degrees of generality consists of

- Rady and Sandmann (1994), Rady (1997), Miltersen et al. (1997), Goldys (1997), Ingersoll (1997), and Lipton (2001) who study the case when X is bounded and thus a true martingale for Foreign Exchange and Interest Rate markets;
- Albanese et al. (2001) and Lipton (2002), who derive prices for European calls on X in the case when P has one or two real roots;

- Zühlsdorff (2001, 2002), Andersen (2011), and Chibane (2011) who compute prices for European calls and puts in the general case.

Most of this research focuses on deriving analytic expressions for the valuation of European-style contingent claims. We refer to Andersen (2011) for the precise formulas of European calls and puts. In the following sections, we shall derive purely probabilistic methods to easily compute the price of any possibly path-dependent contingent claim.

2 Connection to Wiener process

Bluman (1980, 1983), Carr et al. (2002) and Lipton (2001) prove that the partial differential equations (PDEs) corresponding to the class of QNV models are the only parabolic PDEs that can be reduced to Wiener processes via a certain set of transformations. In this section, we derive a probabilistic equivalent while, in particular, paying attention to the issues of strict local martingality. More precisely, we shall see that if one starts on a Wiener space equipped with a Brownian motion W , is allowed

1. to stop W at a stopping time τ ,
2. transform W^τ by a strictly increasing smooth function f , and
3. to change the probability measure with a density that only depends on W_T^τ on \mathcal{F}_T ,

then $f(W^\tau)$ is under the new measure a QNV process up to time τ , given that it is a local martingale.

We start with a simple technical observation:

Lemma 1 (Necessary condition for path-independence of integrals). *Let τ be a stopping time of the form*

$$\tau := \inf\{t \geq 0 : W_t \notin (x, z)\}, \quad \inf \emptyset := \infty$$

for some $x \in [-\infty, 0)$ and $z \in (0, \infty]$. Let $h : (x, z) \rightarrow \mathbb{R}$ denote a continuous function such that

$$\int_0^t h(W_s) ds = \tilde{h}(t, W_t) \tag{2}$$

on $\{\tau > t\}$ for some measurable function $\tilde{h} : [0, \infty) \times (x, z) \rightarrow (-\infty, \infty)$. Then, $h(\cdot) \equiv C$ for some $C \in \mathbb{R}$.

Proof. Assume that (2) holds but h is not a constant. Then, there exists some $\epsilon > 0$ and some $y \in (x + \epsilon, z - \epsilon)$ such that $|h(y) - h(0)| = 5\epsilon$, without loss of generality $y \in (0, z - \epsilon)$. Now, define $\tilde{y} := \inf\{y \in [0, z) : |h(y) - h(0)| \geq 5\epsilon\}$. Assume, again without loss of generality, $h(0) = 0$ and $h(\tilde{y}) = 5\epsilon > 0$. Observe that there exists some $\delta \in (0, \min(-x, \epsilon))$ such that $|h(y)| < \epsilon$ for all y with $|y| < \delta$ and $h(y) > 4\epsilon$ for all y with $|y - \tilde{y}| < \delta$.

Now, fix $T > 0$ and observe that the event $\{|W_t(\omega)| < \delta \text{ for all } t \in [0, T]\}$ has positive probability and thus, $|\tilde{h}(x)| < T\epsilon$ for all x with $|x| < \delta$. However, the event

$$\{-\delta \leq W_t(\omega) \leq \tilde{y} + \delta \text{ for all } t \in [0, T]\} \cap \{\tilde{y} - \delta \leq W_t(\omega) < \tilde{y} + \delta \text{ for all } t \in [0.1T, 0.8T]\}$$

$$\bigcap \{|W_t(\omega)| < \delta \text{ for all } t \in [0.9T, T]\}$$

also has positive probability implying $\tilde{h}(x) \geq (-5 \cdot 0.1 + 4 \cdot 0.7 - 5 \cdot 0.1 - 1 \cdot 0.1)\epsilon T > \epsilon T$ for all x with $|x| < \delta$, leading directly to a contradiction. \square

The next lemma relates the solution of three ODEs to each other:

Lemma 2 (Three ODEs). *Fix $C, d, f_0 \in \mathbb{R}$ and $\mu_-, \mu_+ \in [-\infty, \infty]$ with $\mu_- < 0 < \mu_+$ and let $\mu : (\mu_-, \mu_+) \rightarrow \mathbb{R}$ solve the ODE*

$$\mu'(x) - \mu^2(x) = C, \quad \mu(0) = \mu_0. \quad (3)$$

Then $f, g : (\mu_-, \mu_+) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} f(x) &:= d \int_0^x \exp\left(2 \int_0^y \mu(z) dz\right) dy + f_0, \\ g(x) &:= \exp\left(-\int_0^x \mu(z) dz\right) \end{aligned} \quad (4)$$

solve the ODEs

$$f'(x) = e_1 f^2(x) + e_2 f(x) + e_3, \quad f(0) = f_0; \quad (5)$$

$$-g''(x) = Cg(x), \quad g(0) = 1, \quad g'(0) = -\mu_0, \quad (6)$$

respectively, for appropriate $e_1, e_2, e_3 \in \mathbb{R}$.

Proof. The ODE in (6) can be checked easily. For (5), first consider the case $\mu_0^2 = -C$. The uniqueness of solutions to (3) yields $\mu \equiv \mu_0$; see for example Section 8.2 in Hirsch and Smale (1974). If $\mu_0 \neq 0$ then $f(x) := d(\exp(2\mu_0 x - 1)/(2\mu_0) + f_0$ and if $\mu_0 = 0$ then $f(x) = dx + f_0$ satisfy both (5). Consider now the case $\mu_0^2 \neq -C$ and observe that by a similar uniqueness argument again this implies $\mu^2(x) \neq -C$ for all $x \in (\mu_-, \mu_+)$. We obtain that

$$\log\left(\frac{\mu'(x)}{\mu_0^2 + C}\right) = \log\left(\frac{\mu^2(x) + C}{\mu_0^2 + C}\right) = \int_0^x \frac{2\mu(z)\mu'(z)}{\mu^2(z) + C} dz = 2 \int_0^x \mu(z) dz.$$

Therefore, $f(x) = d(\mu(x) - \mu_0)/(\mu_0^2 + C) + f_0$, which clearly satisfies (5). \square

The next lemma provides the full set of solutions for the ODEs in (5) and (6):

Lemma 3 (Solutions of ODEs). *For any e_1, e_2, e_3, f_0 the ODE in (5) has a unique solution f defined in a neighborhood around zero (up to an explosion) and μ defined by*

$$\mu(x) := \frac{1}{2} \frac{f''(x)}{f'(x)} = e_1 f(x) + \frac{e_2}{2} \quad (7)$$

satisfies (3) with $\mu_0 = e_1 f_0 + e_2/2$ and $C = e_1 e_3 - e_2^2/4$. Moreover, with $P(z) = e_1 z^2 + e_2 z + e_3$, the solutions for g as defined in (4) and f are

- if $e_1 = 0$:

$$g(x) = \exp(-e_2 x/2),$$

$$f(x) = \left(f_0 + \frac{e_3}{e_2}\right) \exp(e_2 x) - \frac{e_3}{e_2} \text{ or } f(x) = e_3 x + f_0 \quad (\text{if } e_2 = 0);$$

- if P has a double root (at $r = -e_2/2e_1$) or equivalently if $C = 0$ and $e_1 \neq 0$:

$$g(x) = 1 - \mu_0 x = 1 - \left(e_1 f_0 + \frac{e_2}{2}\right) x,$$

$$f(x) = (f_0 - r) \frac{1}{1 - \mu_0 x} + r = \left(f_0 + \frac{e_2}{2e_1}\right) \frac{1}{1 - (e_1 f_0 + e_2/2)x} - \frac{e_2}{2e_1};$$

- if P has two roots (at $r_1 = (-e_2/2 - \sqrt{-C})/e_1$ and $r_2 = (-e_2/2 + \sqrt{-C})/e_1$) or equivalently if $C < 0$ and $e_1 \neq 0$

– and additionally $f_0 \in (r_1 \wedge r_2, r_1 \vee r_2)$ or equivalently $\mu_0 \in (-\sqrt{-C}, \sqrt{-C})$:

$$g(x) = \frac{\cosh(\sqrt{-C}x + c)}{\cosh(c)} = \frac{\cosh\left(\sqrt{e_2^2/4 - e_1 e_3}x + c\right)}{\cosh(c)},$$

$$f(x) = \frac{-\sqrt{-C}}{e_1} \tanh\left(\sqrt{-C}x + c\right) - \frac{e_2}{2e_1} = \frac{-\sqrt{e_2^2/4 - e_1 e_3}}{e_1} \tanh\left(\sqrt{e_2^2/4 - e_1 e_3}x + c\right) - \frac{e_2}{2e_1}$$

with

$$c := \operatorname{artanh}\left(\frac{-\mu_0}{\sqrt{-C}}\right) = \frac{1}{2} \log\left(\frac{\sqrt{-C} - \mu_0}{\sqrt{-C} + \mu_0}\right);$$

– and additionally $f_0 \notin [r_1 \wedge r_2, r_1 \vee r_2]$ or equivalently $\mu_0 \notin [-\sqrt{-C}, \sqrt{-C}]$:

$$g(x) = \frac{\sinh(\sqrt{-C}x + c)}{\sinh(c)} = \frac{\sinh\left(\sqrt{e_2^2/4 - e_1 e_3}x + c\right)}{\sinh(c)},$$

$$f(x) = \frac{-\sqrt{-C}}{e_1} \coth\left(\sqrt{-C}x + c\right) - \frac{e_2}{2e_1} = \frac{-\sqrt{e_2^2/4 - e_1 e_3}}{e_1} \coth\left(\sqrt{e_2^2/4 - e_1 e_3}x + c\right) - \frac{e_2}{2e_1}$$

with

$$c := \operatorname{arcoth}\left(\frac{-\mu_0}{\sqrt{-C}}\right) = \frac{1}{2} \log\left(\frac{-\sqrt{-C} + \mu_0}{\sqrt{-C} + \mu_0}\right);$$

– and additionally $f_0 \in \{r_1 \wedge r_2, r_1 \vee r_2\}$ or equivalently $\mu_0 \in \{-\sqrt{-C}, \sqrt{-C}\}$:

$$g(x) = \exp(-\mu_0 x),$$

$$f(x) \equiv f_0;$$

- if P has no real roots and is not constant or equivalently if $C > 0$:

$$g(x) = \frac{\cos(\sqrt{C}x + c)}{\cos(c)} = \frac{\cos(\sqrt{e_1 e_3 - e_2^2/4}x + c)}{\cos(c)},$$

$$f(x) = \frac{-\sqrt{C}}{e_1} \tan(-\sqrt{C}x + c) - \frac{e_2}{2e_1} = \frac{-\sqrt{e_1 e_3 - e_2^2/4}}{e_1} \tan(-\sqrt{e_1 e_3 - e_2^2/4}x + c) - \frac{e_2}{2e_1}$$

with $c := \arctan(-\mu_0/\sqrt{C})$.

Proof. The existence and uniqueness of solutions of the ODE in (5) follow from standard arguments in the theory of ODEs; see Section 8.2 in Hirsch and Smale (1974). The fact that μ satisfies the corresponding ODE and the other statements can be easily checked. \square

Later on when studying changes of numéraires, we shall need the following observation:

Lemma 4 (Reciprocal of a solution f). *For any solution f of (5), define $\hat{f} := 1/f$. Then, \hat{f} solves the Riccati equation*

$$\hat{f}'(x) = -e_3 \hat{f}^2(x) - e_2 \hat{f}(x) - e_1, \quad \hat{f}(0) = f_0^{-1}. \quad (8)$$

If $\hat{\mu}$ and \hat{g} are defined as in (7) and (4), then $\hat{g} = g f / f_0$ (up to an explosion of f), where g is as in (4). In particular, if y is an explosion time of f then $\lim_{x \rightarrow y} g(x) f(x)$ exists and is real.

Proof. Observe that $\hat{f}'(x) = -f'(x)/f^2(x)$, which directly yields (8). Now, $\hat{\mu} = \mu - f'/f$ implies

$$\hat{g}(x) := \exp\left(-\int_0^x \hat{\mu}(z) dz\right) = g(x) \exp\left(\int_0^x \frac{f'(z)}{f(z)} dz\right) = \frac{g(x)f(x)}{f_0}$$

This completes the proof since \hat{g} again satisfies an ODE as in (6) and thus has no explosion. \square

The description of the solutions of the ODEs in (5) and (6) is the fundamental step to prove the following theorem which characterizes QNV processes as the only local martingales that can be simulated from stopped Brownian motion by a certain set of transformations:

Theorem 1 (QNV process and Brownian motion). *Consider the canonical filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where Ω is the space of continuous paths W taking values in \mathbb{R} and \mathbb{P} is the Wiener measure. Denote by $f : \mathbb{R} \rightarrow [-\infty, \infty]$ a measurable function that is three times differentiable in $A := \{x \in \mathbb{R} : |f(x)| < \infty\}$, which is assumed to be open, and satisfies $f'(x) \neq 0$ for all $x \in A$ and by $\tilde{g} : [0, \infty) \times \mathbb{R} \rightarrow [0, \infty)$ a measurable function with $\tilde{g}(0, 0) = 1$. Define the stopping time τ by*

$$\tau := \inf\{t \geq 0 : W_t \notin A\}, \quad \inf \emptyset := \infty$$

and assume that the process $Z = \{Z_t\}_{t \geq 0}$ defined by $Z_t := \tilde{g}(t \wedge \tau, W_t^\tau)$ is a martingale. Consider the process $Y = (Y_t)_{t \geq 0}$, defined by $Y_t := f(W_t^\tau)$ for all $t > 0$, and assume that it is a local martingale under the probability measure \mathbb{Q} defined on \mathcal{F}_t by $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = Z_t$.

Then, Y under \mathbb{Q} satisfies

$$dY_t = (e_1 Y_t^2 + e_2 Y_t + e_3) dB_t \quad (9)$$

for all $t \geq 0$ for some $e_1, e_2, e_3 \in \mathbb{R}$ and \mathbb{Q} -Brownian motion B ; to wit, Y is a QNV process. Furthermore, the corresponding density process Z is of the form $Z_t = \exp(Ct/2)g(W_t^\tau)$ for all $t \geq 0$ for the functions g explicitly computed in Lemma 3, where $C \in \mathbb{R}$ is as in Lemma 3. Conversely, for any $e_1, e_2, e_3 \in \mathbb{R}$ there exist f, \tilde{g}, \mathbb{Q} with the corresponding \mathbb{Q} -Brownian motion B as above so that (9) for $Y_t = f(W_t)$ holds.

Proof. By the martingale representation theorem (see for example Theorem III.4.33 of Jacod and Shiryaev, 2003) and by the fact that zero is an absorbing point of the martingale Z there exists some progressively measurable process $\tilde{\mu} = (\tilde{\mu}_t)_{t \geq 0}$, such that the dynamics of Z can be described as $dZ_t = -Z_t \tilde{\mu}_t dW_t$ for all $t \geq 0$. Then, by Girsanov's theorem, the process $B = (B_t)_{t \geq 0}$ defined as $B_t := W_t + \int_0^t \tilde{\mu}_s ds$ for all $t \geq 0$ is a \mathbb{Q} -Brownian motion; see page 192 in Karatzas and Shreve (1991).

Itô's formula yields

$$dY_t = df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt = f'(W_t)dB_t + \left(\frac{1}{2}f''(W_t) - f'(W_t)\tilde{\mu}_t\right)dt \quad (10)$$

for all $t < \tau$. Thus, by the uniqueness of the Doob-Meyer decomposition (see Theorem III.16 in Protter, 2003), we have $\tilde{\mu}_t = \mu(W_t)$ for all $t < \tau$, where we have defined

$$\mu(x) := \frac{1}{2} \frac{f''(x)}{f'(x)}$$

for all $x \in A$. By the assumption on f , the function μ is differentiable in A .

Then, another application of Itô's formula yields

$$\log(\tilde{g}(W_t)) = - \int_0^t \mu(W_s) dW_s - \frac{1}{2} \int_0^t \mu^2(W_s) ds = -\eta(W_t) + \frac{1}{2} \int_0^t (\mu'(W_s) - \mu^2(W_s)) ds,$$

on $\{\tau > t\}$, where $\eta : A \rightarrow (-\infty, \infty)$ is defined as $\eta(x) = \int_0^x \mu(y) dy$ for all $x \in A$, which we from now on assume to be convex, without loss of generality. Then, Lemma 1 yields that μ satisfies the ODE of (3) in A . Lemma 2 implies that f solves the ODE of (5); in conjunction with (10), this yields (9). Applying Lemma 2 moreover gives that Z is necessarily of the claimed form.

Finally, Lemma 3 shows that for any quadratic polynomial P there exists a differentiable function f such that $f' = P(f)$. Thus, to prove the last part of the statement it is sufficient to show that the corresponding changes of measure exist, that is, to show that the functions g computed in Lemma 3 satisfy $\mathbb{E}^\mathbb{P}[g(W_t^\tau) \exp(Ct/2)] = 1$ for the corresponding $C \in \mathbb{R}$. This is clear in the cases $e_1 = 0$ and $C \geq 0$ since $g(W_t^\tau) \exp(Ct/2)$ then is either a stopped \mathbb{P} -(geometric) Brownian motion or a bounded \mathbb{P} -martingale by Itô's formula. By remembering the definitions $\sinh(x) := (\exp(x) - \exp(-x))/2$ and $\cosh(x) := (\exp(x) + \exp(-x))/2$ and thus being able to write $g(W_t^\tau) \exp(Ct/2)$ as the sum of two true martingales in the case of two roots, we conclude. \square

The next corollary concludes this section by illustrating how expectations of path-dependent functionals in QNV models can be computed. Here and in the following, we shall always assume $\infty \cdot \mathbf{1}_A(\omega) = 0$ if $\omega \notin A$ for any $A \in \mathcal{F}$, for sake of notation.

Corollary 1 (Computation of expectations in QNV models). *Fix $T > 0$ and let $h : C([0, T], \mathbb{R}) \rightarrow [0, \infty]$ denote any nonnegative measurable function of continuous paths. Then, there exist functions f, g , a real number C , and a stopping time τ as in Theorem 1 such that*

$$\begin{aligned}\mathbb{E}[h((Y_t)_{t \in [0, T]})] &= \mathbb{E} \left[h((f(B_t))_{t \in [0, T]}) \mathbf{1}_{\{\tau > T\}} \exp \left(\frac{CT}{2} \right) g(B_T) \right], \\ \mathbb{E}[h((X_t)_{t \in [0, T]})] &= \mathbb{E} \left[h((f(B_t^S))_{t \in [0, T]}) \mathbf{1}_{\{\tau > T \wedge S\}} \exp \left(\frac{C(T \wedge S)}{2} \right) g(B_T^S) \right],\end{aligned}$$

where as before Y (X) denotes a (stopped) QNV process, $B = (B_t)_{t \geq 0}$ a standard Brownian motion, and S the first hitting time of zero by the process $f(B_t)$.

Proof. The statement follows directly from Theorem 1, first for bounded h and then for any nonnegative h by taking the limit, after observing that $\{\tau > T\} = \{g(B_T^\tau) > 0\}$ and that $g(B_T^\tau) = g(B_T)$ on the event $\{\tau > T\}$. \square

Let us use the notation $\underline{B}_T := \min_{t \in [0, T]} B_t$ and $\bar{B}_T := \max_{t \in [0, T]} B_t$. Then, the event $E_1 := \{\tau > T\}$ can be represented as events of the form $\{f^- < \underline{B}_T < \bar{B}_T < f^+\}$ for some $f^-, f^+ \in [-\infty, \infty]$. More precisely, with the notation of Lemma 3, assuming, without loss of generality, that $\mu_0 \geq 0$,

- if $e_1 = 0$ or $C < 0$ and $\mu_0 \in [-\sqrt{-C}, \sqrt{-C}]$ then $E_1 = \emptyset$;
- if $e_1 \neq 0$ and $C = 0$ then $E_1 = \{\bar{B}_T < 1/\mu_0\}$;
- if $e_1 \neq 0$ and $C < 0$ and $\mu_0 \notin [-\sqrt{-C}, \sqrt{-C}]$ then $E_1 = \{\bar{B}_T < -c/\sqrt{-C}\}$;
- if $C > 0$ then $E_1 = \{(c - \pi/2)/\sqrt{C} < \underline{B}_T < \bar{B}_T \leq (c + \pi/2)/\sqrt{C}\}$.

The events $E_2 := \{\tau > T \wedge S\}$ have the same representation with B always replaced by B^S . It can easily be checked that $E_2 = \emptyset$ if the polynomial corresponding to X has a root greater than f_0 . These considerations illustrate that for any QNV model, quantities of the form $\mathbb{E}[\tilde{h}(Y_T)]$ or $\mathbb{E}[\tilde{h}(X_T)]$ can easily be computed by using only the joint distribution of Brownian motion together with its running minimum and maximum.

3 Connection to geometric Brownian motion

We now focus on the case when Y is a QNV process with a polynomial P that has exactly two real roots r_1, r_2 . For that, we parameterize P as $P(z) = e_1(z - r_1)(z - r_2)$ for some $e_1 \in \mathbb{R} \setminus \{0\}$ and $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$.

In the following, we shall connect the dynamics of a QNV process to geometric Brownian motion. This link has been established for the case $y_0 \in (r_1, r_2)$ in Rady (1997).

Theorem 2 (QNV process and stopped geometric Brownian motion). *Fix $T > 0$. Let $h : C([0, T], \mathbb{R}) \rightarrow [0, \infty]$ denote any measurable nonnegative function of continuous paths. If Y is a QNV process with polynomial $P(z) = e_1(z - r_1)(z - r_2)$ for some $e_1 \in \mathbb{R} \setminus \{0\}$ and $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, then*

$$\mathbb{E}[h((Y_t)_{t \in [0, T]})] = \frac{y_0 - r_1}{r_2 - r_1} \mathbb{E} \left[h \left(\left(\frac{r_2 - r_1 Z_t}{1 - Z_t} \right)_{t \in [0, T]} \right) \mathbf{1}_{\{Z_T \neq 1\}} (1 - Z_T) \right] \quad (11)$$

where Z denotes a (possibly negative) geometric Brownian motion stopped upon hitting 1 with $Z_0 = (y_0 - r_2)/(y_0 - r_1)$ and

$$\frac{dZ_t}{Z_t} = e_1(r_2 - r_1)dB_t \quad (12)$$

for all $t \geq 0$, where $B = (B_t)_{t \geq 0}$ denotes some (possibly stopped) Brownian motion.

Proof. Define two processes $M = \{M_t\}_{t \geq 0}$ and $N = \{N_t\}_{t \geq 0}$ by

$$M_t := \frac{x_0 - r_1}{r_2 - r_1} \cdot (1 - Z_t) \text{ and } N_t := \frac{r_2 - r_1 Z_t}{1 - Z_t} \mathbf{1}_{\{Z_t < 1\}}$$

Then M is a nonnegative martingale started in one and thus, defines a new measure $\tilde{\mathbb{Q}}$ by $d\tilde{\mathbb{Q}} = M_T d\mathbb{Q}$. It is sufficient to show that N under $\tilde{\mathbb{Q}}$ has the dynamics of Y . First, we observe that MN is a \mathbb{Q} -martingale up to the stopping time S^M , defined as the first hitting time of zero by M . Therefore, N is a $\tilde{\mathbb{Q}}$ -local martingale; see also Proposition 2.8(ii) in Carr et al. (2012). Furthermore, we observe that by Itô's formula

$$d\langle N \rangle_t = \left(\frac{r_2 - r_1}{(1 - Z_t)^2} \right)^2 d\langle Z \rangle_t = e_1^2 \frac{(r_2 - r_1)^4 Z_t^2}{(1 - Z_t)^4} dt = e_1^2 (N_t - r_1)^2 (N_t - r_2)^2 dt.$$

Thus, N is a local martingale under $\tilde{\mathbb{Q}}$ with the same quadratic variation as X . This implies that the distribution of X under \mathbb{Q} is the same as the one of N under $\tilde{\mathbb{Q}}$, which proves the statement. \square

We observe that the process Z is negative if and only if $y_0 \in (r_1, r_2)$, exactly the case treated by Rady (1997). In that case Z is not stopped as it never hits 1. It is also exactly this case when Y is a true martingale; compare Proposition 1. Indeed, using $h(\omega) = \omega_T$ in (11) shows that

$$\begin{aligned} \mathbb{E}[Y_T] &= \frac{y_0 - r_1}{r_2 - r_1} \mathbb{E}[(r_2 - r_1 Z_T) \mathbf{1}_{\{Z_T \neq 1\}}] = \frac{y_0 - r_1}{r_2 - r_1} (\mathbb{E}[(r_2 - r_1 Z_T)] - (r_2 - r_1) \mathbb{Q}(Z_T = 1)) \\ &= y_0 - (y_0 - r_1) \mathbb{Q}(Z_T = 1) \end{aligned}$$

which equals y_0 if and only if $y_0 \in (r_1, r_2)$.

Remark 1 (Connections to the reciprocal of the three-dimensional Bessel process). It is well-known that the reciprocal \tilde{Y} of the three-dimensional Bessel process defined by the dynamics $d\tilde{Y}_t = -\tilde{Y}_t^2 dB_t$ for all $t \geq 0$ is distributed as the reciprocal of a stopped (upon hitting zero) Brownian motion started in 1 after a change of numéraire. To see the connection to the last proposition, consider the case of a QNV process Y with polynomial $P(z) = z(z - r_2)$ where $0 < r_2 < 1 = y_0$.

Then, as r_2 tends to zero, the dynamics of Y resemble more and more the ones of \tilde{Y} . Now, define Z as in (12) stopped when hitting 1 and observe that

$$\frac{1 - Z_t}{r_2} = \frac{1}{r_2} \left(1 - (1 - r_2) \exp \left(r_2 B_t - \frac{r_2^2 t}{2} \right) \right) = 1 - B_t + O(r_2)$$

by a Taylor series expansion. Therefore, as r_2 gets closer to zero, the argument of h on the right-hand side of (11) tends to the reciprocal of a Brownian motion started in 1, exactly as we would expect. \square

4 Closedness under changes of measure

We recall the process X defined as $X := Y^S$, where S denotes the first hitting time of zero by Y . In particular, X is a nonnegative \mathbb{Q} -local martingale. In Appendix A, we discuss the construction of a probability measure $\hat{\mathbb{Q}}$ with the local density process X . The measure $\hat{\mathbb{Q}}$ has the financial interpretation of the change of risk-neutral dynamics if the numéraire changes. In this section, we study this change of measure for stopped QNV processes and observe that the class of stopped QNV processes is stable under changes of numéraires, a feature which makes QNV processes attractive as models for foreign exchange rates.

We start with a simple observation that is related to the statement of Lemma 4:

Lemma 5 (Roots of quadratic polynomial). *We consider the polynomial $P(z) := e_1 z^2 + e_2 z + e_3$ of Section 1 and its counterparty $\hat{P}(z) := -z^2 P(1/z) = -e_3 z^2 - e_2 z - e_1$. They satisfy the duality relations*

- *P only has complex roots if and only if \hat{P} only has complex ones;*
- *P has zero as root if and only if $\hat{P}(z) = 0$ describes a linear equation, and vice versa;*
- *P has zero as a double root if and only if \hat{P} is constant, and vice versa.*

Proof. The statement follows from simple considerations such as that if $r \in \mathbb{R} \setminus \{0\}$ is a root of P then $1/r$ is a root of \hat{P} . \square

Lemma 5 yields the next proposition, which shows that the QNV property is stable under a change of numéraire. In this context, we remind the reader of (8), which we shall utilize on in the next section.

Proposition 2 (Closedness under change of numéraire). *QNV processes are closed under a change of numéraire. That is, the process $\hat{X} := 1/X$ is again a stopped QNV process, now under the probability measure $\hat{\mathbb{Q}}$, which is defined as in Theorem 3 as the Föllmer measure corresponding to X as underlying, with polynomial $P^{\hat{X}}(z) := -e_3 z^2 - e_2 z - e_1$. More precisely, under a change of numéraire*

- (i) *QNV processes with complex roots are closed;*
- (ii) *QNV processes with two real non-zero roots are closed;*

- (iii) QNV processes with a single root at zero are dual to shifted geometric Brownian motion;
- (iv) Non-constant QNV processes with a double root at zero are dual to (constantly time-changed) Brownian motion.

Proof. The reciprocal \widehat{X} of X is by Theorem 3 a $\widehat{\mathbb{Q}}$ -local martingale. Using a localization argument and computing its quadratic variation then yields that it is a QNV process, now with polynomial $P^{\widehat{X}}$; see also Subsection 2.3 and, in particular, Proposition 4 in Carr et al. (2012). The statements in (i) to (iv) follow from Lemma 5. \square

We now are ready to give a simple proof of Proposition 1, stated in the introduction:

Proof. (of Proposition 1) For the stopped QNV process X , strict local martingality is equivalent to $\widehat{\mathbb{Q}}(\widehat{X} = 0) > 0$ for the probability measure $\widehat{\mathbb{Q}}$ and the stopped $\widehat{\mathbb{Q}}$ -QNV process \widehat{X} introduced in Proposition 2; see Theorem 3. We now observe that $\widehat{\mathbb{Q}}(\widehat{X} = 0) > 0$ is equivalent to $e_1 \neq 0$ (as otherwise 0 is a root of $P^{\widehat{X}}$) together with the condition that all roots of $P^{\widehat{X}}$ are greater or equal than $1/y_0$. The last condition is equivalent to the condition that all roots of P are smaller than y_0 ; see Lemma 5.

If $e_1 = 0$ then Y is either constant or Brownian motion (if $e_2 = 0$) or $\widetilde{Y} := Y + e_3/e_2$ is geometric Brownian motion (if $e_2 \neq 0$). In both cases, it is clear that Y is a true martingale. If y_0 lies between two roots of P then Y is bounded, thus a martingale. For the reverse direction, assume that Y is a martingale and that $e_1 \neq 0$. Then, there exists a root $r \geq y_0$ of P since otherwise $X = Y^S$ is a strict local martingale. Denote the second root of P by \widetilde{r} and define the QNV process $\widetilde{Y} := r - Y$ with polynomial $P^{\widetilde{Y}}(z) = -e_1 z(z - (r - \widetilde{r}))$. It is clear that \widetilde{Y} is again a martingale and thus, by the same argument $r - \widetilde{r} \geq \widetilde{Y}_0 = r - y_0$, which yields the statement. \square

5 Semi-static hedging

In the following, we present an interesting symmetry that can be applied for the semi-static replication of barrier options in certain parameter setups as we discuss below.

Proposition 3 (Symmetry). *We fix $T > 0$ and assume that X is a stopped QNV process with a polynomial of the form $P(z) = az^2/L + e_2z + aL$ and $x_0 = L$ for some $L > 0$ and $a, e_2 \in \mathbb{R}$. Let $h : [0, \infty] \rightarrow [0, \infty]$ denote some measurable nonnegative function satisfying $h(0) = 0$ and $h(\infty) \in \mathbb{R}$. We then have the equivalence*

$$h\left(\frac{X_T}{L}\right) \in \mathcal{L}^1(\mathbb{Q}) \Leftrightarrow h\left(\frac{L}{X_T}\right) \frac{X_T}{L} \in \mathcal{L}^1(\mathbb{Q})$$

and the identity

$$\mathbb{E}\left[h\left(\frac{X_T}{L}\right)\right] = \mathbb{E}\left[h\left(\frac{L}{X_T}\right) \frac{X_T}{L}\right]. \quad (13)$$

In particular, by using $h(x) = \mathbf{1}_{x>0}\mathbf{1}_{x<\infty}$, we obtain

$$\mathbb{E}[X_T] = L\mathbb{Q}(X_T > 0)$$

and by replacing $h(x)$ by $h(x)\mathbf{1}_{x>1}$,

$$\mathbb{E} \left[h \left(\frac{X_T}{L} \right) \mathbf{1}_{\{X_T > L\}} \right] = \mathbb{E} \left[h \left(\frac{L}{X_T} \right) \frac{X_T}{L} \mathbf{1}_{\{X_T < L\}} \right]. \quad (14)$$

Proof. We observe that Z defined via $Z_t := X_t/L$ for all $t \geq 0$ is a stopped QNV process with a polynomial of the form $P(z) = az^2 + e_2z + a$ and $Z_0 = 1$. Thus, we can assume, without loss of generality, that $L = 1$. Now, Theorem 3 yields with $H = h(X_T)/X_T \mathbf{1}_{\{X_T > 0\}}$ that

$$\mathbb{E} [h(X_T)] = \mathbb{E}^{\hat{\mathbb{Q}}} \left[\frac{h(X_T)}{X_T} \right] = \mathbb{E} \left[h \left(\frac{1}{X_T} \right) X_T \right],$$

where the second equality follows from observing that $1/X$ has the same distribution under $\hat{\mathbb{Q}}$ as X has under \mathbb{Q} ; see Proposition 2. This shows (13) and the other parts of the statement follow directly from it. \square

Remark 2 (Alternative proof of Proposition 3). The last statement can also directly be shown without relying on the change of measure technique of Theorem 3. For this, we again assume $L = 1$ and define the sequences of processes X^n ($X^{1/n}$) by stopping X as soon as it hits n ($1/n$) for all $n \in \mathbb{N}$. We then observe that Girsanov's Theorem (Theorem VIII.1.4 of Revuz and Yor, 1999) implies for all $\epsilon > 0$ and all Borel sets $A_\epsilon \subset (\epsilon, \infty)$

$$\mathbb{E} [X_T^n \mathbf{1}_{\{1/X_T^n \in A_\epsilon\}}] = \mathbb{Q} (X_T^{1/n} \in A_\epsilon).$$

Now, we first let n go to ∞ and then ϵ to zero and obtain

$$\mathbb{E} [X_T \mathbf{1}_{\{1/X_T \in A\}}] = \mathbb{Q} (X_T^{1/n} \in A \cap (0, \infty))$$

for all Borel sets A , which again yields (13). \square

Remark 3 (Semi-static hedging). Proposition 3 and, in particular, (14) can be interpreted as the existence of a semi-static hedging strategy for barrier options in the spirit of Bowie and Carr (1994), Carr et al. (1998), and Carr and Lee (2009).

To see this, consider a QNV process X with a polynomial of the form $P(z) = az^2/L + bz + aL$ and $x_0 > L$ for some $L > 0$ and $a, b \in \mathbb{R}$. Consider further a down-and-in barrier option with barrier L and terminal payoff $h(X_T/L)$ if the barrier is hit by X . For a semi-static hedge, at time zero one buys two positions of European claims, the first paying off $h \left(\frac{X_T}{L} \right) \mathbf{1}_{\{X_T \leq L\}}$ and the second paying off $h \left(\frac{L}{X_T} \right) \frac{X_T}{L} \mathbf{1}_{\{X_T < L\}}$. If the barrier is not hit, both positions have zero price at time T . If the barrier is hit, however, one sells the second position and buys instead a third position paying off $h \left(\frac{X_T}{L} \right) \mathbf{1}_{\{X_T > L\}}$. The equality in (14) guarantees that these two positions have the same price at the hitting time. This strategy is semi-static because it only requires trading at maximally two points of time.

Proposition 3, in particular, contains the well-known case of geometric Brownian motion ($a = 0$), where semi-static hedging is always possible. It is an open question to determine more general symmetries than the one of Proposition 3. One difficulty here arises from the lack of equivalence of the two measures \mathbb{Q} and $\hat{\mathbb{Q}}$. However, it is clear that adding an independent change of time to the dynamics of X keeps any existing such symmetry. \square

6 Joint replication and hyperinflation

In this section, we continue with a financial point of view and interpret the probability measure \mathbb{Q} as the unique risk-neutral measure, under which the stopped QNV process X denotes the price of an asset, say, the price of a Euro in Dollars. The probability measure $\hat{\mathbb{Q}}$, introduced Section 4 can then be interpreted as the unique risk-neutral probability measure of a European investor who uses the price of a Euro as a numéraire. To emphasize this point we shall use from now on the notation $\mathbb{Q}^{\$} := \mathbb{Q}$ and $\mathbb{Q}^{\epsilon} := \hat{\mathbb{Q}}$.

In Carr et al. (2012), we suggest to use a novel pricing operator for contingent claims to restore put-call parity, even in the case if the underlying is a strict local martingale. It is defined as the minimal superreplicating cost of a contingent claim under both measures $\mathbb{Q}^{\$}$ and \mathbb{Q}^{ϵ} ; observe that these two probability measures are not equivalent if X is a strict local martingale. More precisely we suggest as a price $p^{\$}$ (denoted here in Dollars) of a contingent claim the minimal amount of Dollars to construct an admissible trading strategy with a wealth process that dominates the payoff of the claim $\mathbb{Q}^{\$}$ - and \mathbb{Q}^{ϵ} -almost surely. In this section, we shall provide a representation of this joint replicating price.

Formally, we fix a finite time horizon T and assume the canonical path space with X denoting the paths. Moreover, we assume a complete market with a bond paying zero interest rate and the risky asset X modeling the price of one Euro in Dollars and following the dynamics of a stopped QNV process with polynomial P .

We then describe a contingent claim by a pair $D = (D^{\$}, D^{\epsilon})$ of random variables measurable with respect to the sigma-algebra generated by X up to time T . Here, the first component of D represents the claim's (random) payoff denoted in Dollars at time T as seen under the measure $\mathbb{Q}^{\$}$ and the second its (random) payoff denoted in Euros at time T as seen under the measure \mathbb{Q}^{ϵ} . In particular, it always holds that $D^{\epsilon} = D^{\$}/X_T$ on the event $\{0 < X_T < \infty\}$, which corresponds exactly to the states of the world that both measures $\mathbb{Q}^{\$}$ and \mathbb{Q}^{ϵ} can “see.”

In the following corollary, where we compute the minimal joint replicating price of such a claim D under both $\mathbb{Q}^{\$}$ and \mathbb{Q}^{ϵ} , we denote, by \mathbb{P} , the probability measure under which the canonical process has the representation $X = f(W^S)$ for some Brownian motion W :

Corollary 2 (Minimal joint replicating price in a QNV model). *For the minimal joint replicating price $p^{\$}(D)$ of a contingent claim $D = (D^{\$}, D^{\epsilon})$ under $\mathbb{Q}^{\$}$ and \mathbb{Q}^{ϵ} , we have the identity*

$$p^{\$}(D) = \mathbb{E}^{\mathbb{P}} \left[(D^{\$} \mathbf{1}_{\{\tau > S \wedge T\}}) \exp(C(T \wedge S)) g^{\$}(W_T^S) \right] + x_0 \mathbb{E}^{\mathbb{P}} \left[(D^{\epsilon} \mathbf{1}_{\{\tau \leq S \wedge T\}}) \exp(C(T \wedge \tau)) g^{\epsilon}(W_T^{\tau}) \right], \quad (15)$$

for τ , \mathbb{P} , C , $g^{\$} \equiv g$ as in Theorem 1 and g^{ϵ} similarly but corresponding to the stopped QNV process \hat{X} of Proposition 2.

Proof. Theorem 2 in Carr et al. (2012) proves the representation

$$p^{\$}(D) = \mathbb{E}^{\mathbb{Q}^{\$}} [D^{\$}] + x_0 \mathbb{E}^{\mathbb{Q}^{\epsilon}} [D^{\epsilon} \mathbf{1}_{\{1/X_T = 0\}}]$$

By Corollary 1, the first term on the right-hand side corresponds to the first term on the right-hand side of (15). For the second term, we observe that $\hat{X} = 1/X$ is a stopped QNV process for some polynomial $P^{\hat{X}}$ under \mathbb{Q}^{ϵ} by Proposition 2. We thus can conclude by the same line of thought using Lemma 4 after observing that τ and S change places and that the real number C does not change if one transforms \hat{X} instead of X . \square

We emphasize the symmetry of τ and S that we relied on in the proof of the corollary. The stopping time τ is the first time $f(W)$ hits infinity and $\hat{f}(W) = 1/f(W)$ hits zero and the stopping time S satisfies the converse statement.

We also remark that the probability measure \mathbb{P} can be interpreted as a physical measure, under which hyperinflations occur with positive probability. Thus, $f(W^S)$ can be used to model an exchange rate that allows (under \mathbb{P}) for hyperinflations in either Euros or Dollars; then, $p^\$$ represents the minimal replicating cost (in Dollars) for a claim that pays $D^\$$ Dollars if no hyperinflation of the Dollar occurs and that pays D^ϵ Euros if the Dollar hyperinflates, corresponding to the Dollar price of a Euro being infinity. For a deeper discussion on this interpretation, we refer to Carr et al. (2012).

The expression in (15) symbolically reduces to

$$p^\$(D) = \mathbb{E}^\mathbb{P} \left[\tilde{D} \exp(C(T \wedge S \wedge \tau)) g^\$(W_T^{\tau \wedge S}) \right], \quad (16)$$

where $\tilde{D} = D^\$ = D^\epsilon \cdot f(W_T^{\tau \wedge S})$ and all multiplications of 0 with ∞ are interpreted in the sense of (15); see also Lemma 4.

As a brief illustration of the last corollary, let us consider the minimal joint replicating price of one Euro in Dollars, to wit, $D = (X_T, 1)$. From (15) we obtain

$$\begin{aligned} p^\$(D) &= \mathbb{E}^\mathbb{P} \left[\exp(C(T \wedge S \wedge \tau)) \left((f(W_T^{\tau \wedge S}) \mathbf{1}_{\{\tau > S \wedge T\}}) g^\$(W_T^{\tau \wedge S}) + x_0 \mathbf{1}_{\{\tau \leq S \wedge T\}} g^\epsilon(W_T^{\tau \wedge S}) \right) \right] \\ &= x_0 \mathbb{E}^\mathbb{P} \left[\exp(C(T \wedge S \wedge \tau)) \left(\mathbf{1}_{\{\tau > S \wedge T\}} g^\epsilon(W_T^{\tau \wedge S}) + \mathbf{1}_{\{\tau \leq S \wedge T\}} g^\epsilon(W_T^{\tau \wedge S}) \right) \right] \\ &= x_0 \mathbb{E}^\mathbb{P} \left[\exp(C(T \wedge S \wedge \tau)) g^\epsilon(W_T^{\tau \wedge S}) \right] = x_0, \end{aligned}$$

where we have used the representation $g^\epsilon = f g^\$ / x_0$ of Lemma 4. Thus, the minimal replicating cost for one Euro is exactly x_0 Dollars, exactly what we hoped for. We remark that the symbolic representation of (16) with $\tilde{D} = f(W_T^{\tau \wedge S})$ directly yields the same statement, too.

A Föllmer's measure

In this appendix, we present a result that follows directly from a slightly more general formulation in our companion paper Carr et al. (2012). For any nonnegative continuous local martingale X , the theorem yields the existence of a new probability measure $\hat{\mathbb{Q}}$, which can be interpreted as the measure corresponding to a change of numéraire. For example, if X represents the price of Euros in Dollars and if \mathbb{Q} is a risk-neutral measure for prices denoted in Dollars, then $\hat{\mathbb{Q}}$ corresponds to the measure under which asset prices denoted in Euros (instead of Dollars) follow local martingale dynamics. We remind the reader that we defined $\infty \cdot \mathbf{1}_A(\omega) = 0$ if $\omega \notin A$ for any $A \in \mathcal{F}$.

Theorem 3 (Generalized change of measure). *Let X be a nonnegative \mathbb{Q} -local martingale. Fix $T > 0$ and consider the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{Q}})$ with $\tilde{\Omega}$ the set of paths $\tilde{\omega} : [0, T] \rightarrow [0, \infty]$ getting absorbed when hitting either zero or infinity, $\tilde{\mathcal{F}} = \sigma(\{\tilde{\omega} \in \tilde{\Omega}\})$, $\tilde{\mathbb{F}}$ the right-continuous modification of the filtration generated by the canonical process, and $\tilde{\mathbb{Q}}$ the probability measure induced by the embedding $\omega \in \Omega \mapsto X(\omega) \in \tilde{\Omega}$.*

Then, there exists a unique probability measure $\hat{\mathbb{Q}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ with corresponding expectation operator $\mathbb{E}^{\hat{\mathbb{Q}}}$, such that

$$\mathbb{E}^{\hat{\mathbb{Q}}} [H \mathbf{1}_{\{X_T < \infty\}}] = \frac{\mathbb{E}^{\tilde{\mathbb{Q}}} [H \mathbf{1}_{\{X_T > 0\}} X_T]}{x_0}$$

for any $\tilde{\mathcal{F}}$ -measurable random variable $H \in [0, \infty]$, where we denote, with a slight misuse of notation, the canonical process on $\tilde{\Omega}$ again by X .

Furthermore, $1/X$ is a $\hat{\mathbb{Q}}$ -local martingale and X is a strict \mathbb{Q} -local martingale if and only if $\hat{\mathbb{Q}}(X_T = \infty) > 0$.

It is important to point out that if X is a strict local martingale or has positive probability to hit zero then the constructed probability measure $\hat{\mathbb{Q}}$ and the original probability measure \mathbb{Q} are not equivalent.

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